

**LOUIS BACHELIER**  
**ON THE CENTENARY OF *THÉORIE DE LA SPÉCULATION*\***

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## 1. CENTENARY OF MATHEMATICAL FINANCE

The date March 29, 1900, should be considered as the birthdate of mathematical finance. On that day, a French postgraduate student, Louis Bachelier, successfully defended at the Sorbonne his thesis *Théorie de la Spéculation*. As a work of exceptional merit, strongly supported by Henri Poincaré, Bachelier's supervisor, it was published in *Annales Scientifiques de l'École Normale Supérieure*, one of the most influential French scientific journals.

This pioneering analysis of the stock and option markets contains several ideas of enormous value in both finance and probability. In particular, the theory of Brownian motion, one of the most important mathematical discoveries of the twentieth century, was initiated and used for the mathematical modeling of price movements and the evaluation of contingent claims in financial markets.

The thesis of Louis Bachelier, together with his subsequent works, deeply influenced the whole development of stochastic calculus and mathematical finance. As a testimony of his great contribution, the newly created international Bachelier Finance Society is named after him. The centenary of the famous thesis is widely celebrated as a landmark

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event in the history of modern science. On this occasion we present to the readers' attention a brief account of our research on Louis Bachelier. In spite of his remarkable contributions, he remained in obscurity for decades—one of the most mysterious figures in mathematics of the twentieth century, about whom only a few facts could be found in the literature. Recently there has been enormous public interest in his scientific biography, resulting from the amazingly fast development of mathematical finance in the last two decades and a deeper understanding of the fundamental role of Brownian motion.

We believe that our short note brings some new light on Louis Bachelier, as a person, a mathematician, and a philosopher.

11 March 1870	Louis Jean-Baptiste Alphonse Bachelier is born in Le Havre
October 1888	Graduates from secondary school at Caen
11 January 1889	Father's death
7 May 1889	Mother's death
1889–1891	Bachelier is the head of <i>Bachelier fils</i>
1891–1892	Military service
1892	Student at Sorbonne
October 1895	Bachelor in sciences at Sorbonne
July 1897	Certificate in mathematical physics
29 March 1900	Bachelier defends his thesis, <i>Théorie de la Spéculation</i>
1909–1914	Free lecturer at Sorbonne
1912	Publication of <i>Calcul des Probabilités</i>
1914	Publication of <i>Le Jeu, la Chance et le Hasard</i>
9 September 1914	Drafted as a private in the French army
31 December 1918	Back from the army
10 December 1919	A member of the French Mathematical Society
1919–1922	Assistant professor in Besançon
14 September 1920	Marries Augustine Jeanne Maillot; she died soon
1922–1925	Assistant professor in Dijon
1925–1927	Associate professor in Rennes
January 1926	Blackballed in Dijon
1 October 1927	Professor in Besançon
1937	Professor Emeritus
1 October 1937	Retirement
1941	The last publication
28 April 1946	Louis Bachelier dies in Saint-Servan-sur-Mer; and is buried in Sanvic near Le Havre
1996	The Bachelier Finance Society is founded

## 2. BACHELIER'S EARLY YEARS

Louis Bachelier was born into a respected bourgeois family known in Le Havre for its cultural and social traditions. His father, Alphonse Bachelier, was not only a wine merchant but also the vice-consul of Venezuela at Le Havre and an amateur scientist. His mother, Cecile Fort-Meu, was a banker's daughter; his grandfather, an important person in the financial business, was known also as the author of poetry books. Unfortunately, just after Louis graduated from the secondary school at Caen, earning the French bachelor

degree *baccalauréat es sciences*, his parents died and he had to interrupt his studies to continue his father's business and take care of his sister and three-year-old brother.

This dramatic event had far-reaching consequences for his academic career; in particular, they explain why Bachelier did not follow any *grande école* with the French scientific elite, a weak point in his curriculum. Nevertheless, as the head of the family enterprise (officially registered as *Bachelier fils*) he became acquainted with the world of financial markets, and one can find some hints of personal experiences in his works.

Military service soon followed, bringing further delay of his studies (an enormous handicap in a forming mathematician!). Finally, in 1892 Louis arrived in Paris to continue his university education at the Sorbonne. Scarce information is available about these years, though it is known that he followed lectures of Paul Appell, Joseph Boussinesq, and Henri Poincaré and that, apparently, he was not among the best students (his marks in mathematics in the 1895 register were largely below those of his classmates Langevin and Liénard). But in spite of the four-year interruption, Bachelier's development as a scientist was fast enough and he wrote his celebrated thesis, *Théorie de la Spéculation*, on the application of probability to stock markets. This was historically the first attempt to use advanced mathematics in finance and witnessed the introduction of Brownian motion. In accordance with the tradition of the era, he also defended a second thesis on a subject chosen by the faculty, namely, on mechanics of fluids. Its title may reflect Bachelier's educational background: *Résistance d'une masse liquide indéfinie pourvue de frottements intérieurs, régis par les formules de Navier, aux petits mouvements variés de translation d'une sphère solide, immergée dans cette masse et adhérente à la couche fluide qui la touche*.

### 3. THE THESIS AND POINCARÉ'S REPORT

The first part of Bachelier's thesis, *Théorie de la Spéculation*, contains a detailed description of products available at that time in the French stock market, such as forward contracts and options. Their specifications were quite different from the corresponding products in the American market (see comments in Cootner 1964); for example, all payments were related to a single date and one had no need to think about the discounting or change of numéraire. After financial preliminaries, Bachelier begins the mathematical modeling of stock price movements and formulates the principle that "the expectation of the speculator is zero." Obviously, he understands here by expectation the conditional expectation given the past information. In other words, he implicitly accepts as an axiom that the market evaluates assets using a martingale measure. The further hypothesis is that the price evolves as a continuous Markov process, homogeneous in time and space. Bachelier shows that the density of one-dimensional distributions of this process satisfies the relation known now as the Chapman–Kolmogorov equation and notes that the Gaussian density with the linearly increasing variance solves this equation. The question of the uniqueness is not discussed, but Bachelier provides some further arguments to confirm his conclusion. He arrives at the same law by considering the price process as a limit of random walks. Bachelier also observes that the family of distribution functions of the process satisfies the heat equation: the probability diffuses or "radiates."

The results in these dozen pages are spectacular, but this is not the end. The model is applied to calculate various option prices. Having in mind American and path-dependent options, Bachelier calculates the probability that the Brownian motion does not exceed a fixed level and finds the distribution of the supremum of the Brownian motion.

One hundred years after the publication of the thesis it is quite easy to appreciate the importance of Bachelier's ideas. The thesis can be viewed as the origin of mathematical finance and of several important branches of stochastic calculus such as the theory of Brownian motion, Markov processes, diffusion processes, and even weak convergence in functional spaces. Of course, the reasoning was not rigorous but it was, on the intuitive level, basically correct. This is really astonishing, because at the beginning of the century the mathematical foundations of probability did not exist. A. Markov started his studies on what are now called Markov chains only in 1906, and the concept of conditional expectations with respect to an arbitrary random variable or  $\sigma$ -algebra was developed only in the 1930s.

Poincaré's report (signed also by P. Appell and J. Boussinesq) is a remarkable document showing that Bachelier's thesis was highly appreciated by the outstanding mathematician. We have included the full report as an Appendix to this paper. The report contains a deep analysis not only of the mathematical results but also an insight into market laws. In contrast to the legend that the evaluation note "*honorable*" means somehow that the examiners were skeptical about the thesis, it seems that it was the highest note which could be awarded for a thesis that was essentially outside mathematics and that had a number of arguments far from being rigorous. The excellent note was usually assigned to memoirs containing the solution to a challenging problem in a well-established mathematical discipline. In our opinion, the report shows clearly that Henri Poincaré was an extremely attentive and benevolent reader, and his mild criticism was positive. The expressed regret that Bachelier did not study in detail the discovered relationship of stochastic processes with equations in partial derivatives can be interpreted that he was really intrigued, seeing here further perspectives. Poincaré's report and the conclusion to publish the thesis in the most prestigious journal of that period contradict what some have labeled as "the disappointing note *honorable*." One could guess that Bachelier was not awarded the note "*très honorable*" because of a weaker presentation of his second thesis (but the corresponding report of P. Appell is very positive: "Bachelier has a good command of works of M. Boussinesq on movements of a sphere in a fluid...").

Needless to say, the innovative ideas of Bachelier were much above the prevailing level of existing financial theory. And they certainly were noticed. In the book by de Montessus (1908) containing several chapters on applications of probability to insurance, artillery, etc., an entire chapter is devoted to probabilistic methods in finance and based on Bachelier's thesis.

#### 4. FURTHER STUDIES

Louis Bachelier remained quite active in the period from 1900 to 1914. He continued to develop the mathematical theory of diffusion processes in a series of papers published in reputed French journals. In his memoir of 1906 he defined new classes of stochastic processes, which are now called processes with independent increments and Markov processes, and he derived the distribution of the Ornstein–Uhlenbeck process (see Jacobsen 1996). His approach can be considered as the development of the theory of stochastic differential equations using the language and concepts of gambling. Two functions playing essential roles in the paper, "relative expectation" and "relative instability," can be identified with the drift and diffusion coefficients of the modern vocabulary.

We could not find any traces of his employment during the first decade. However, he quite regularly received scholarships to continue his studies. During the period 1909 to

1914 Bachelier gave lectures at Sorbonne as “free professor” (he was paid starting only from 1913). In particular, he gave the lecture course “Probability Calculus with Applications to Financial Operations and Analogies with Certain Questions from Physics.”

Apparently, Bachelier was aware of the importance of his contributions. He wrote in his curriculum vitae (“Notice” of 1924) that his book *Calcul des Probabilités*, published in 1912, was the first that surpassed the great treatise by Laplace. The efforts to advertise his approach were not always successful; for example, the demand to get fundings for a monograph on his studies in probability was rejected because the committee considered that “the results in the suggested book are not essentially different from those published yet in the journal of M. Jordan and *Annales Scientifiques de l’École Normale Supérieure*.” Nevertheless, in 1914 he published *Le Jeu, la Chance et le Hasard*, a book that had great public success: more than six thousand copies were sold. Bachelier considered that his principal achievement was the systematic use of the concept of continuity in probabilistic modeling: the continuous distributions are the fundamental objects correctly describing the very nature of many random phenomena and not just mathematical inventions simplifying a work with discrete distributions.

In 1914 the Council of the Paris University supported a move to make Bachelier’s appointment permanent, but the war destroyed this plan. World War I began and Bachelier was drafted as a private; he served in the army until December 31, 1918. Subsequently, he obtained his first regular academic position, in Besançon, replacing professor C.-E. Traynard on leave. The return of Traynard in 1922 forced Bachelier to move to Dijon where he replaced R. Baire, and in 1925 he moved to Rennes. At last, in 1927 he was awarded his permanent professorship in Besançon where he worked until his retirement in 1937.

At the end of his academic career Bachelier published the résumés of his early works, trying to attract public attention to his work on the theory of Markov processes. The final chapter of his scientific career was a note in *Comptes-rendus de l’Académie des Sciences* of 1941 on distributions of functionals of Brownian motion. Remarkably, the importance of these results in mathematical finance—for instance, the pricing of barrier options—was revealed only relatively recently (see Geman and Yor 1996).

## 5. BLACKBALLED IN DIJON

One of the most dramatic events in Bachelier’s life was his attempt to get a position in Dijon in 1926.<sup>1</sup> It was really a disgrace that the Council of the Faculty of Sciences classified him as the second candidate (after G. Cerf), a disgrace not because of the result but the pretext: Professor M. Gevrey blamed Bachelier for an error in the paper of 1913 readily confirmed by a letter of Paul Lévy, professor of École Polytechnique. Bachelier was furious. There exists a letter typed by him in which he exposes the details of this sad story:<sup>2</sup>

Due to a sequence of incredible circumstances . . . I have found myself at the age of 56 in a situation worse than I had during the last six years; this is after twenty-six years with the doctor degree, five years of teaching as free professor at Sorbonne, and six years of official replacement of a full professor. My career happened to be blocked in a deplorable and extremely unfair way without a minor reproach which could be given to me.

<sup>1</sup> Not in Geneva in 1927 as was claimed in Bernstein (1992).

<sup>2</sup> Translated by Yuri Kabanov, this letter is held at the Bibliothèque de l’Institut Henri Poincaré, Document 058.

...

The critique of M. Lévy is simply ridiculous: he accuses me of using . . . continuous formulae for the case where they are, in fact, discontinuous, but a large number of experiments allow for an asymptotic approach.

...

M. Lévy noted that the first formula of the accused paper is the formula of Brownian motion. Exactly, but why is it criticized since it is I myself who presents it?

...

M. Lévy pretends not to know my other five large papers published in *Annales de l'École normale* and in *Journal de Mathématiques pures et appliquées* as well as various notes published elsewhere.

He has written a work of 300 pages on probability without even opening my book on the same subject, the book which is, in some respects, above the book by Laplace and which contains a lot of new results.

He does not know my work on scientific philosophy, which has the fourth edition available today (seven thousand copies).

Briefly, M. Lévy did not know my studies, completely, while writing the letter to M. Gevrey.

He did not know also the works of M. Cerf, as he himself confessed. Nevertheless, he recommended, explicitly, M. Cerf for the first rank.

Paul Lévy, apparently, was quite sincere in his report, misled by an erroneous interpretation of Bachelier's notation. The latter defined the Brownian motion as a limit, as  $\tau \rightarrow 0$ , of random functions that are linear on each interval  $[n\tau, (n+1)\tau]$  with derivatives  $v$  or  $-v$  taken at random with equal probabilities. The dependence of  $v$  on  $\tau$  (as  $c\tau^{-1/2}$ ) was omitted but always assumed by Bachelier. Of course, with a constant  $v$  the reasoning cannot be correct and Lévy no doubt had to write the extremely negative conclusion. To his surprise, several years later in the fundamental paper by Kolmogorov, whose mathematical genius he highly appreciated, Lévy found references to Bachelier works, including the thesis. Lévy also made a revision and observed that a number of properties of the Brownian motion had been discovered by Bachelier several decades earlier. To the honor of Paul Lévy, he wrote a letter with apologies and the two reconciled. But Lévy never appreciated the significance of Bachelier's contribution to financial theory. In his notebook<sup>3</sup> with abstracts of the most important works, Lévy gave the following comment on the thesis: "Too much on finance!"

## 6. THE BACHELIER HERITAGE

There is a traditional public view according to which Louis Bachelier was a mathematical prodigy whose remarkable insight had no impact on the development of the theory of stochastic processes, whose results were totally ignored by the probabilistic and financial communities, and who was underestimated by his supervisors, perhaps because the title of his thesis was not attractive to mathematicians. Unfortunately, certain historical research follows this legend. But our study of documents shows that this simplified story is far from being correct.

As we mentioned earlier, his thesis was published in one of the most prestigious mathematical journals upon the recommendation of Poincaré, as well as being published in a separate book. Very quickly, the results appeared in a book on applied probability (de Montessus 1908). This fact is quite remarkable, because in that period there were few

<sup>3</sup> In the personal papers of Paul Lévy kept at the Jussieu Library of Mathematics.

books on probability. He successfully published his *Calcul des Probabilités* (it was not just a coincidence that the format was *in-quarto* as was the famous treatise of Laplace).

Did his ideas influence further studies? Without any doubt our answer is yes. The key paper of Kolmogorov (1931) on diffusion processes seems to be at least partially inspired by Bachelier's work. Roughly speaking, Kolmogorov, who was at this time at the zenith of his mathematical power, just after suggesting the axioms of probability which made it really a mathematical discipline, did the work suggested in Poincaré's comment on the thesis: he developed in full generality the analytical approach for continuous Markov processes.

It is worth noting that in spite of the fact that in modern English textbooks Brownian motion is traditionally referred to as the Wiener process, the original terminology suggested by W. Feller in his famous treatise *An Introduction to Probability Theory and its Applications* (1957) was the *Wiener–Bachelier* process. Another clear message about Bachelier's importance can be found on the very first page of another outstanding book: *Diffusion Processes and their Sample Paths* by K. Ito and H. McKean (1965). One can also find references to Bachelier in the early literature written by economists (see, e.g., Keynes 1921). Bernstein (1992) tells a story about how Bachelier's thesis and books were discovered in the United States by the pioneers of modern financial theory; see also Merton (1995).

## 7. BACHELIER'S PUBLISHED WORKS

### Books

1. *Théorie de la spéculation*, 1900, Gauthier-Villars, 70 pp.
2. *Calcul des probabilités*, 1912, Gauthier-Villars, Tome 1, 516 pp.
3. *Le Jeu, la Chance et le Hasard*, Bibliothèque de Philosophie scientifique, 1914, E. Flammarion, 320 pp.
4. *Les lois des grands nombres du Calcul des Probabilités*, 1937, Gauthier-Villars, v–vii, 36 pp.
5. *La spéculation et le Calcul des Probabilités*, 1938, Gauthier-Villars, v–vii, 49 pp.
6. *Les nouvelles méthodes du Calcul des Probabilités*, 1939, Gauthier-Villars, v–viii, 69 pp.

### Papers

1. Théorie de la spéculation, *Annales Scientifiques de l'École Normale Supérieure*, 1900, pp. 21–86.
2. Théorie mathématique du jeu, *Annales Scientifiques de l'École Normale Supérieure*, 1901, pp. 143–210.
3. Théorie des probabilités continues, *Journal de Mathématiques Pures et Appliquées*, 1906, pp. 259–327.
4. Étude sur les probabilités des causes, *Journal de Mathématiques Pures et Appliquées*, 1908, pp. 395–425.
5. Le problème général des probabilités dans les épreuves répétées, *Comptes-rendus des Séances de l'Académie des Sciences*, Séance du 25 Mai 1908, pp. 1085–1088.
6. Les probabilités à plusieurs variables, *Annales Scientifiques de l'École Normale Supérieure*, 1910, pp. 339–360.

7. Mouvement d'un point ou d'un système matériel soumis à l'action de forces dépendant du hasard, *Comptes-rendus des Séances de l'Académie des Sciences*, Séance du 14 Novembre 1910, présentée par M. H. Poincaré, pp. 852–855.
8. Les probabilités cinématiques et dynamiques, *Annales Scientifiques de l'École Normale Supérieure*, 1913, pp. 77–119.
9. Les probabilités semi-uniformes, *Comptes-rendus des Séances de l'Académie des Sciences*, Séance du 20 Janvier 1913, présentée par M. Appell, pp. 203–205.
10. La périodicité du hasard, *L'Enseignement Mathématique*, 1915, pp. 5–11.
11. Sur la théorie des corrélations, *Comptes-rendus des Séances de la Société Mathématique de France*, Séance du 7 Juillet 1920, pp. 42–44.
12. Sur les décimales du nombre  $\pi$ , *Comptes-rendus des Séances de la Société Mathématique de France*, Séance du 7 Juillet 1920, pp. 44–46.
13. Le problème général de la statistique discontinue, *Comptes-rendus des Séances de l'Académie des Sciences*, Séance du 11 Juin 1923, présentée par M. d'Ocagne, pp. 1693–1695.
14. Quelques curiosités paradoxales du calcul des probabilités, *Revue de Métaphysique et de Morale*, 1925, pp. 311–320.
15. Probabilités des oscillations maxima, *Comptes-rendus des Séances de l'Académie des Sciences*, Séance du 19 Mai 1941, pp. 836–838.

#### APPENDIX: POINCARÉ'S REPORT\*

##### Report on the Thesis of M. Bachelier, March 29, 1900

The subject chosen by M. Bachelier is somewhat removed from those which are normally dealt with by our applicants. His thesis is entitled "Theory of Speculation" and focuses on the application of Probability Theory to the Stock Exchange. First, one might fear that the author has exaggerated the applicability of Probability Theory as has often been done. Fortunately, this is not the case; in his introduction and in the section entitled "Probability in Stock Exchange Operations" he strives to set limits within which one can legitimately apply this type of calculation. He does not exaggerate the range of his results and I do not think that he is deceived by his formulas.

What can one then legitimately conclude in such a field? It is clear, first, that the market prices of various types of operations have to obey certain laws. Thus one could imagine combinations of prices such that one can win with certainty; the author cites some examples of this. It is evident that such combinations will never occur, or if they do, they will not persist. The buyer believes in a probable rise, otherwise he would not buy, but if he buys, it is because someone sells to him, and this seller obviously believes in a probable decline. From this results that the market, considered as a whole, takes the mathematical expectation of all operations and of all combinations of operations to be zero.

What are the mathematical consequences of such a principle? If one supposes that the deviations are not very large, one may assume that the probability of a deviation from the quoted price does not depend on the absolute value of this price. Under these conditions the principle of mathematical expectation suffices to determine the probability law; one obtains Gauss's celebrated law of errors.

\* Poincaré's Report was translated by Selime Baftiri-Balazoski and Ulrich Haussmann.

As this law has been the object of numerous demonstrations which, for the most part, are logically incorrect, one should be cautious and examine closely this proof, or at least it is necessary to state in a precise manner the hypotheses made. Here the hypothesis which must be made is, as I have just said, that the probability of a deviation from the current market price is independent of the absolute value of this price. The hypothesis holds provided that the deviations are not too large. The author states this clearly, without perhaps, emphasizing it as much as he ought to. It is enough, nevertheless, that he has stated it explicitly so that his reasoning is correct.

The manner in which M. Bachelier deduces Gauss's law is very original, and all the more interesting in that his reasoning can be extended with a few changes to the theory of errors. He derives it in a chapter whose title may at first seem strange, for he calls it "Radiation of Probability." In fact, the author makes a comparison with the analytic theory of heat flow. A bit of thought shows that the analogy is real and the comparison is legitimate. The reasoning of Fourier, almost without change, is applicable to this problem so different from the one for which it was originally created. It is regrettable that M. Bachelier did not develop this part of his thesis further. He could have entered into the details of Fourier's analysis. He did, however, say enough about it to justify Gauss's law and to foresee cases where it would no longer hold.

Once Gauss's law is established, one can easily deduce certain consequences susceptible to experimental verification. Such an example is the relation between the value of an option and the deviation from the underlying. One should not expect a very exact verification. The principle of the mathematical expectation holds in the sense that, if it were violated, there would always be people who would act so as to re-establish it and they would eventually notice this. But they would only notice it if the deviations were considerable. The verification, then, can only be gross. The author of the thesis gives statistics where this happens in a very satisfactory manner.

M. Bachelier then examines a problem that at first would seem to give rise to some very complicated calculations. What is the probability that a certain market price be attained before a certain date? In writing the equation of the problem, one is led to a multiple integral in which there are as many  $\int$  signs superimposed as there are days before the date fixed. This equation seems at first intractable. The author solves it by a short, simple and elegant argument; moreover he remarks on the analogy with M. André's reasoning on the ballot problem. But this analogy is not strict enough to diminish in any way the originality of this ingenious artifice. The author uses it with equal success for other similar problems.

In summary, we are of the opinion that there is reason to authorize M. Bachelier to have his thesis printed and to submit it.

Appell            Poincaré            J. Boussinesq

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